Sparse Spanners in Temporal Cliques and Where to Find Them

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Overview

- Short introduction to temporal graphs

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- Spanners in temporal graphs

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- ”How to Reduce Temporal Cliques to Find Sparse Spanners”

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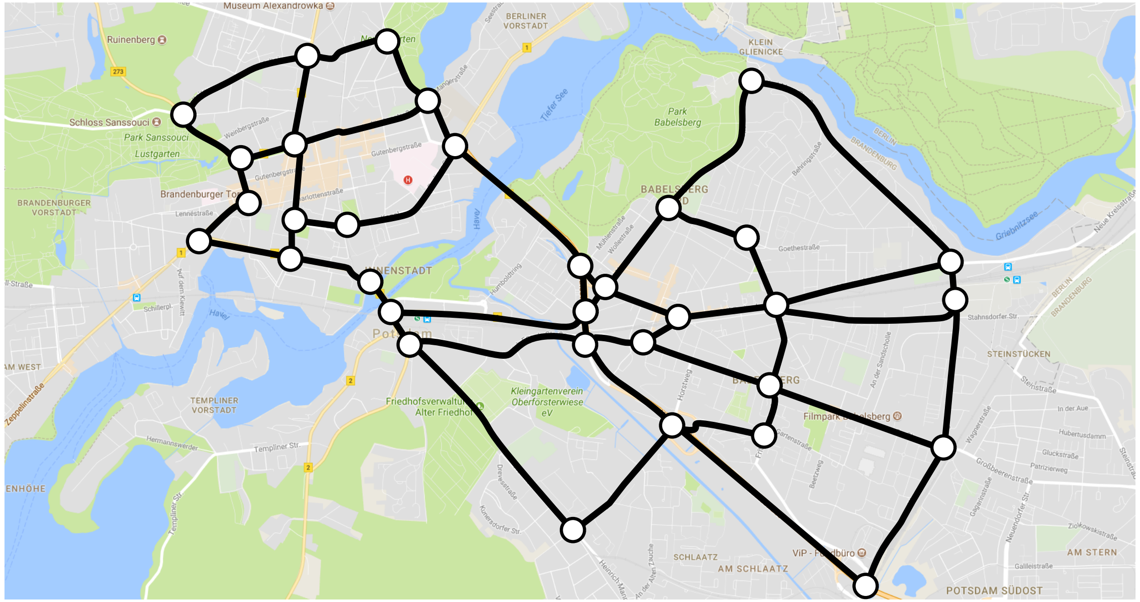
Questions are very welcome!

Temporal Graphs—Motivation

Static Graphs

- Example: road networks



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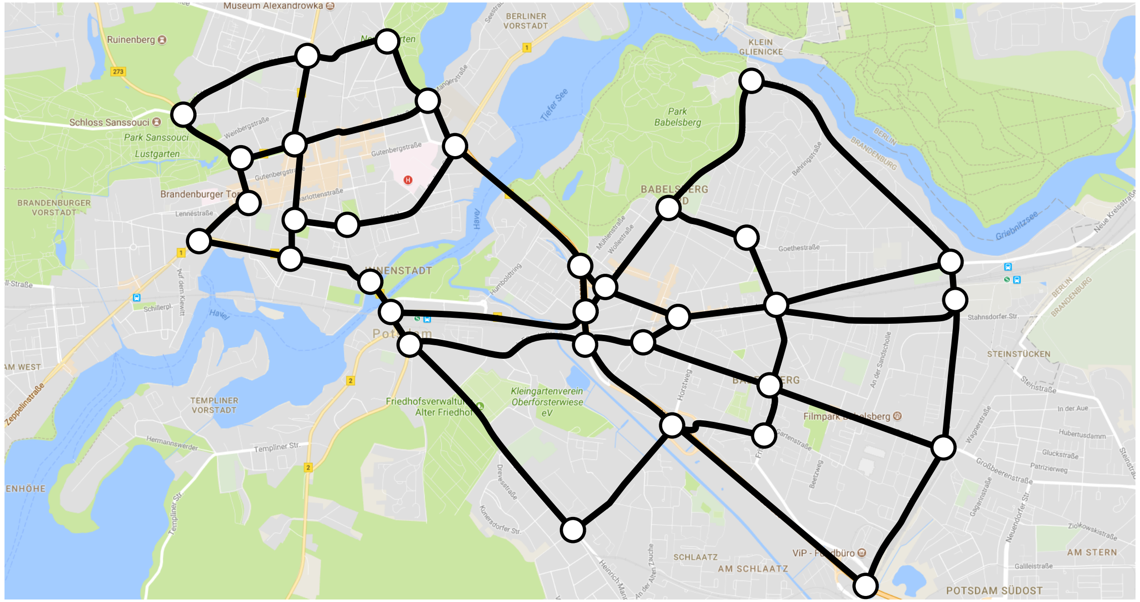
- Example: road networks
- Full algorithmic and graph-theoretic toolkit
- Common problems:
  - Shortest paths
  - Spanning trees
  - Matchings



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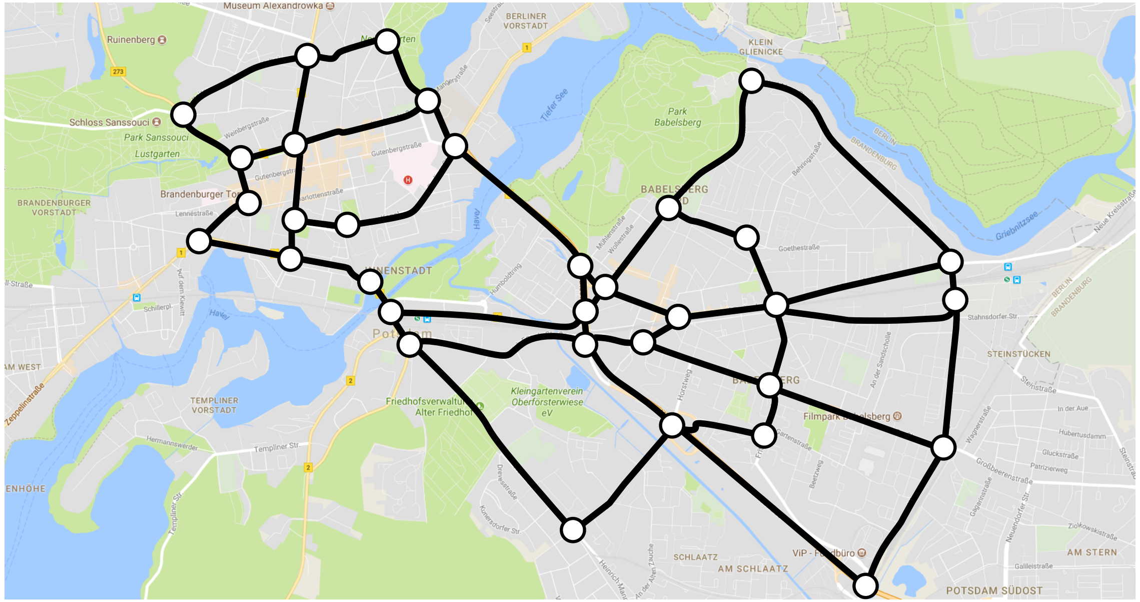
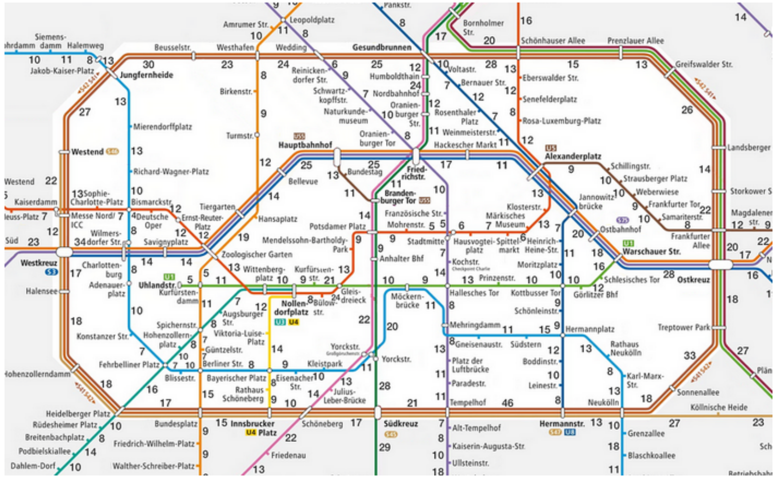


Easy!

Temporal Graphs—Motivation

Social network graph



S-Bahn network of Berlin

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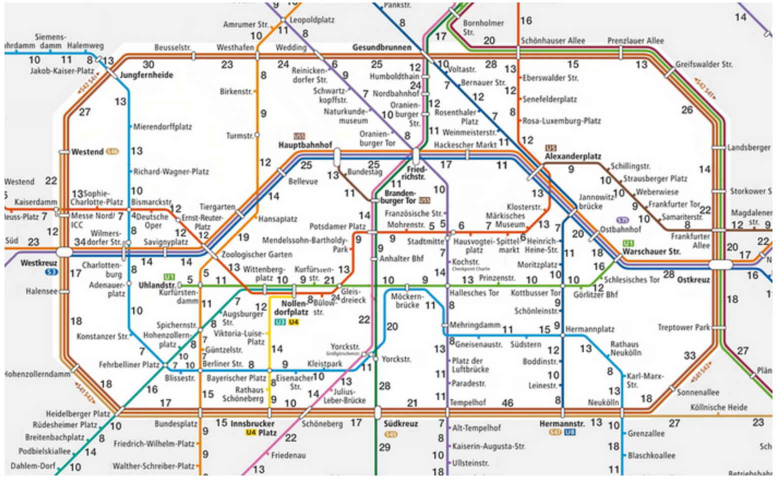

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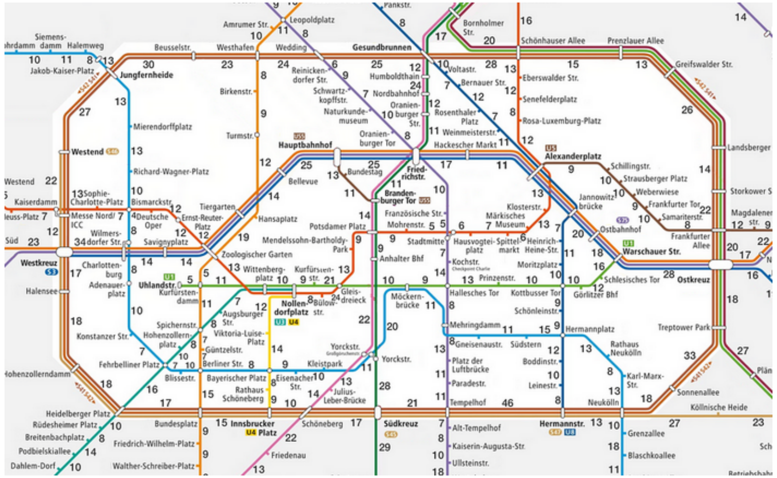

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Dynamic Graphs

- Example: friendship graphs, public transport
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- **Temporal Graphs**: edges available at given timestamps
  - Timestamps known in advance

Static vs Temporal Graphs

Classical Problem

Temporal challenges

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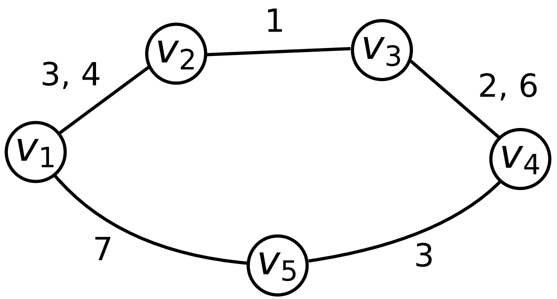
Classical Problem

Shortest Paths

Temporal challenges

Connectivity is not transitive!

$\Rightarrow$ Can be NP-hard



There are paths $v_1 \rightsquigarrow v_2$ and $v_2 \rightsquigarrow v_3$, but not $v_1 \rightsquigarrow v_3$

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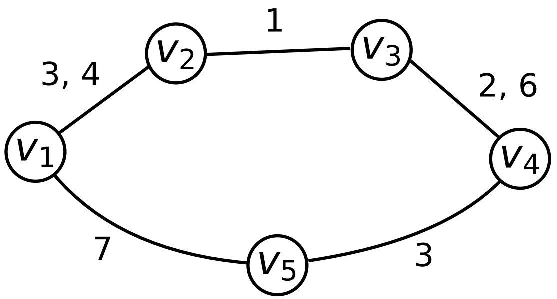

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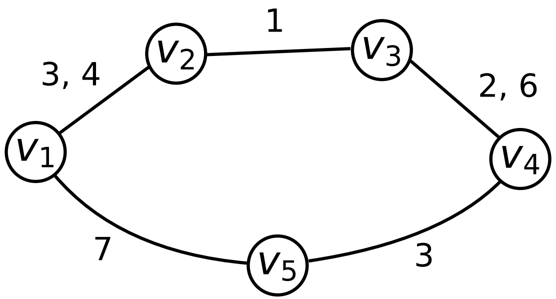
Minimum spanner	Minimum spanners usually not trees $\Rightarrow$ Even quadratic size sometimes!



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Minimum size edge set that yields
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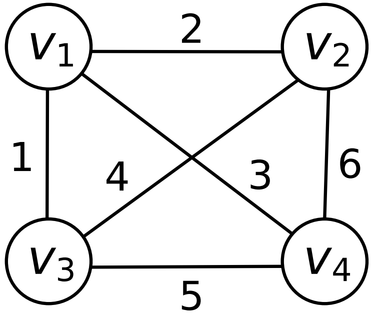

Temporal Cliques

- Underlying graph is a clique
- Each edge has exactly one unique timestamp

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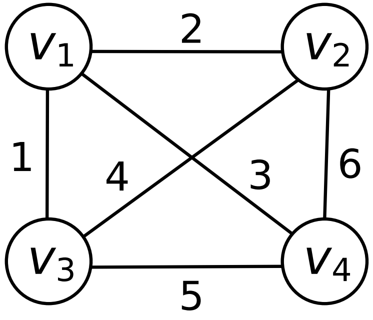

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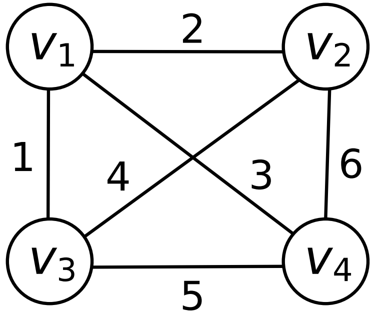
Task: find $\Theta(n)$ sized spanners for temporal cliques!



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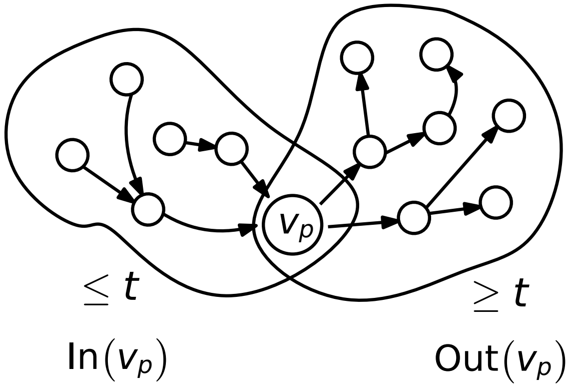

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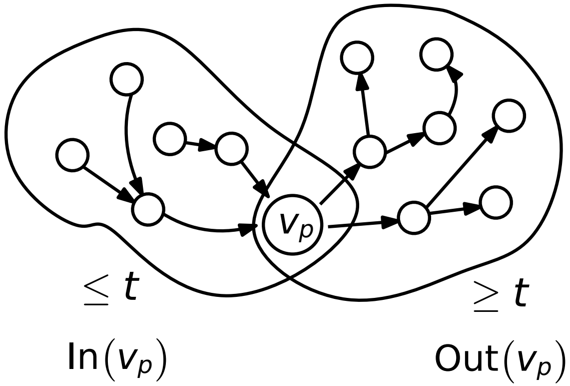

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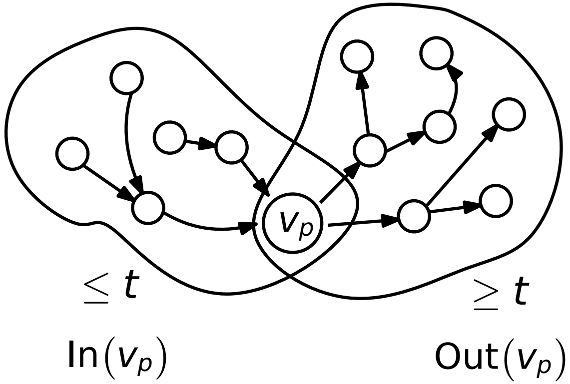

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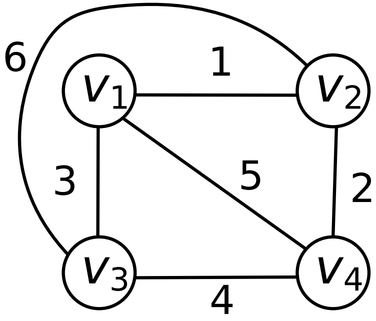

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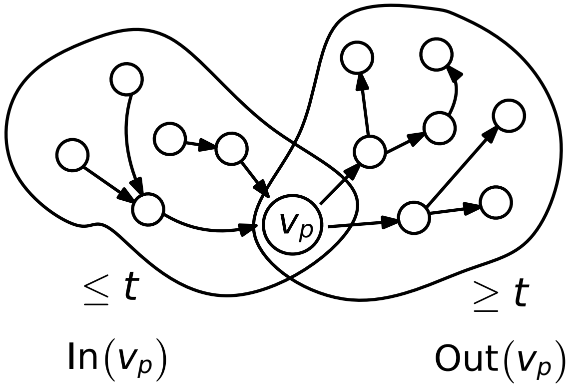

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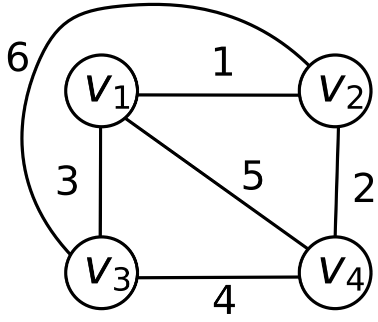

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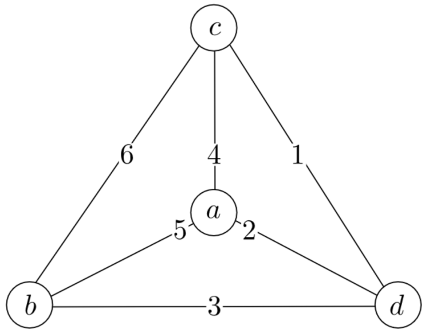
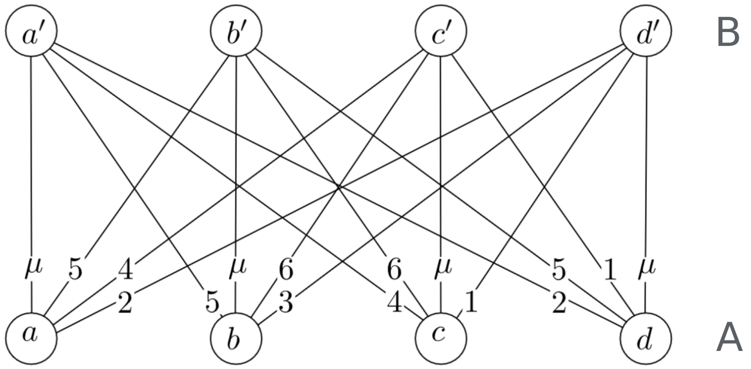
Task: find $\Theta(n)$ sized spanners for **all** temporal cliques!

Temporal Bicliques

- Bicliques are better to work with
- Temporal graph $G = (A \sqcup B, \lambda)$, A to B connectivity



A temporal clique



The corresponding temporal biclique

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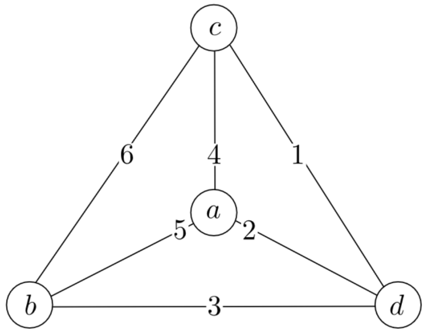
We've proven:

Theorem.
Minimal spanners for bicliques and cliques differ by constant factor



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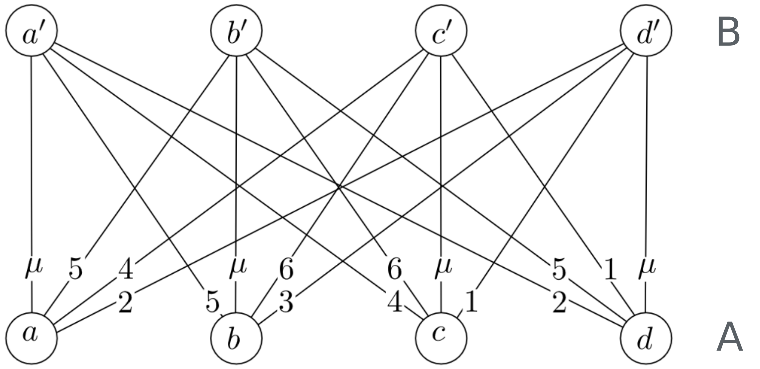

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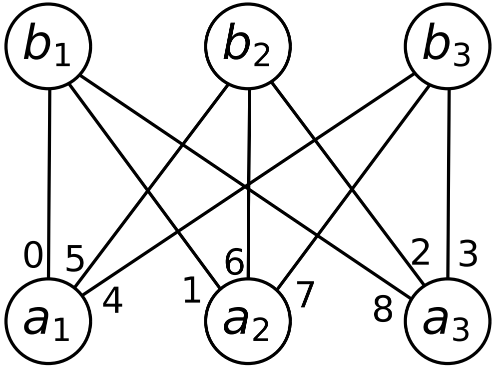

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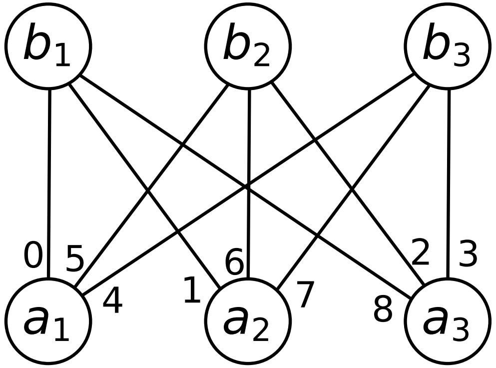
First and last edges form a complete matching



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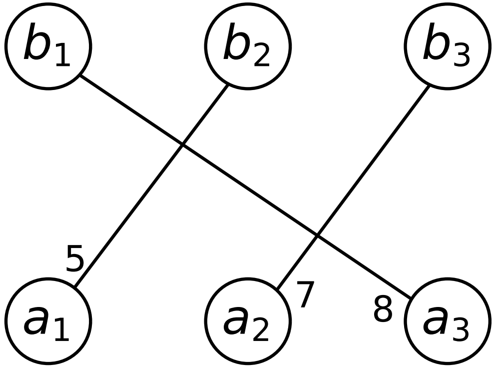

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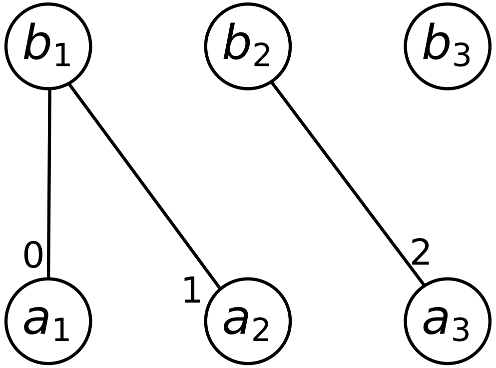

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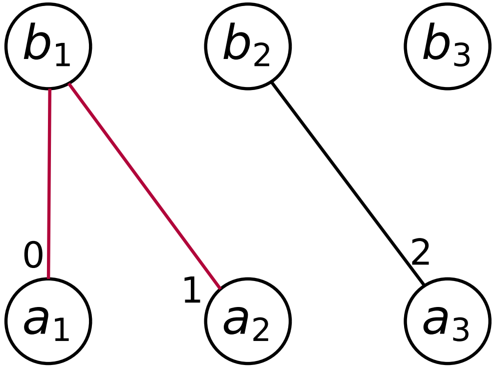

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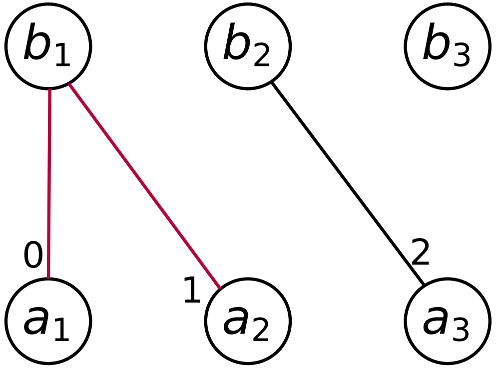

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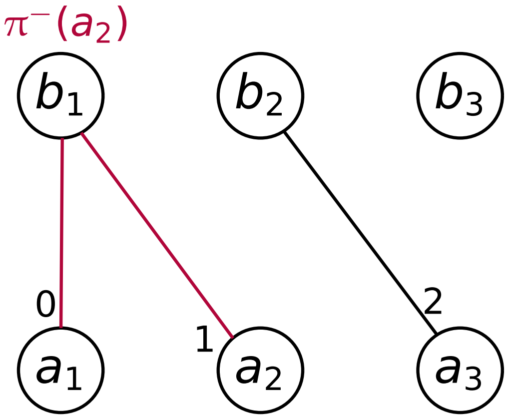

Example of a dismountable vertex

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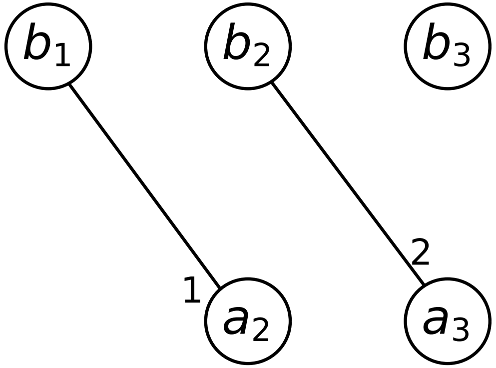

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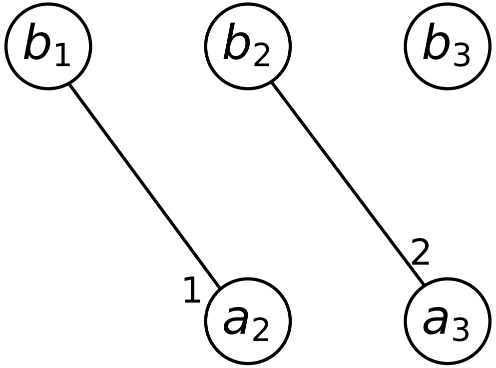

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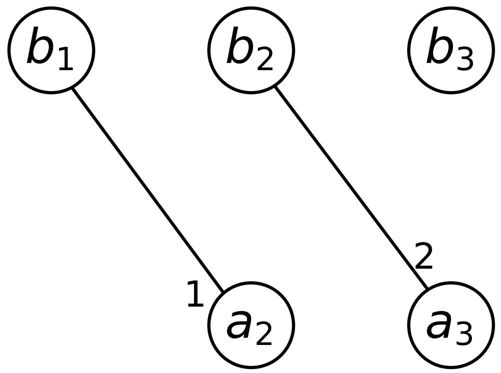
Theorem. Dismount until we have a matching!



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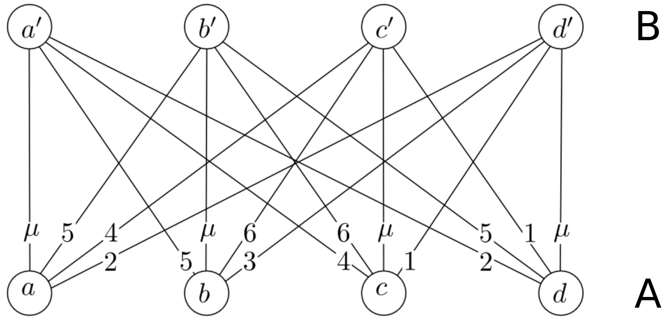
def $\texttt{biSpanner}(D = (A, B, \lambda))$:
  if $|A| = 1$ **then** **return** $E(D)$
  partition A into non-empty A_1, A_2 with $|A_1| = \lfloor |A|/2 \rfloor$
  $D_1 \leftarrow D[A_1 \sqcup B]$, $D_2 \leftarrow D[A_2 \sqcup B]$
  $(D_1^*, S_1^*) \leftarrow \texttt{dismount}(D_1)$, $(D_2^*, S_2^*) \leftarrow \texttt{dismount}(D_2)$
  return $\texttt{biSpanner}(D_1^*) \cup \texttt{biSpanner}(D_2^*) \cup S_1^* \cup S_2^*$

Algorithm for $O(n \log n)$ spanner

Utilize with divide and conquer!



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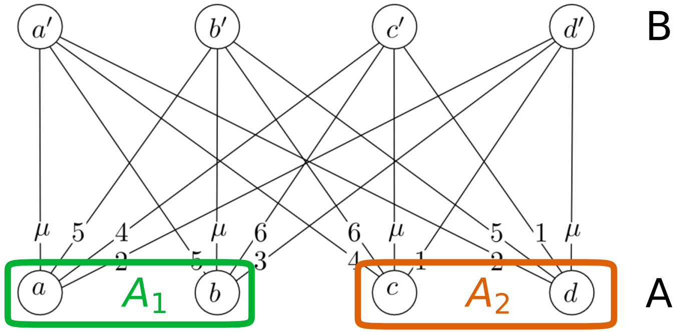

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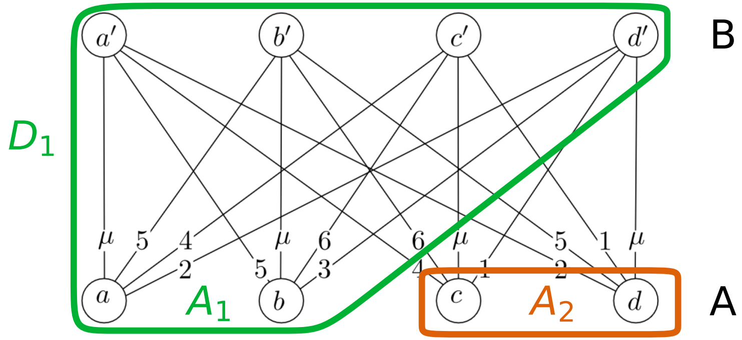

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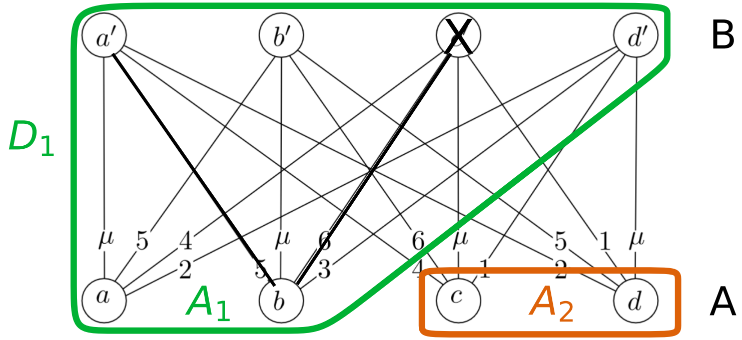

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  if $|A| = 1$ **then return** $E(D)$
  partition A into non-empty A_1, A_2 with $|A_1| = \lfloor |A|/2 \rfloor$
  $D_1 \leftarrow D[A_1 \sqcup B]$, $D_2 \leftarrow D[A_2 \sqcup B]$
→ $(D_1^*, S_1^*) \leftarrow$ dismount(D_1), $(D_2^*, S_2^*) \leftarrow$ dismount(D_2)
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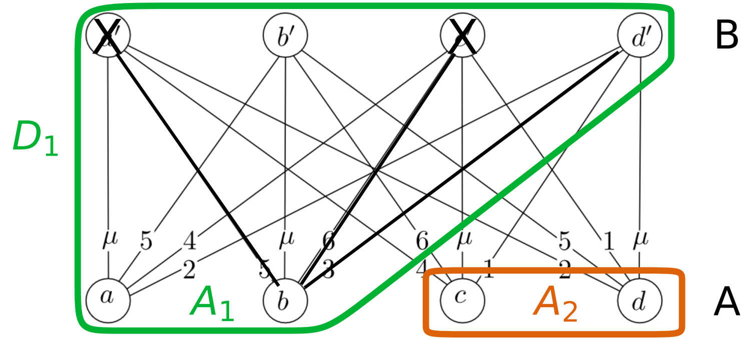
Algorithm for $O(n \log n)$ spanner

Utilize with divide and conquer!



Partition into A_1, A_2

(Extremal) Matchings

- First and last local edges form a matching!
- Take first and last edges into spanner
- Switch roles of A and B
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- $S_1^*, S_2^* \in \Theta(n)$
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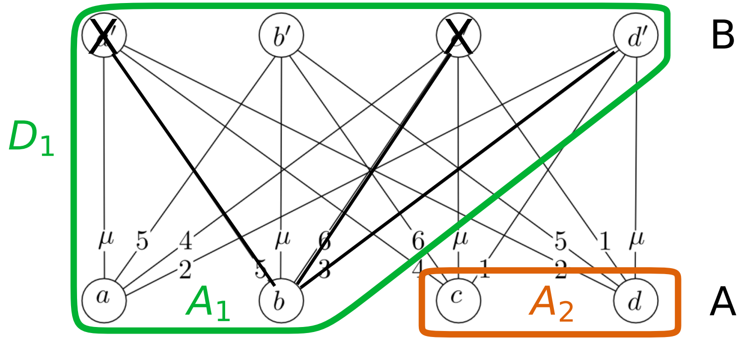

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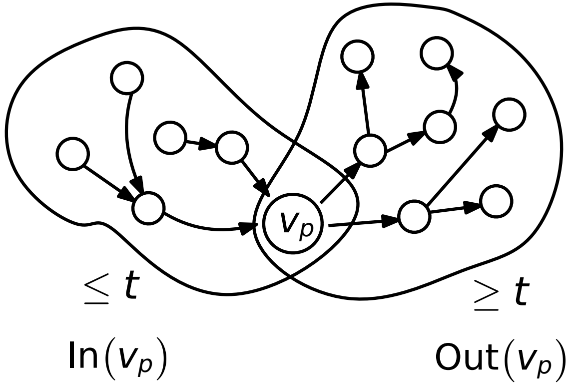

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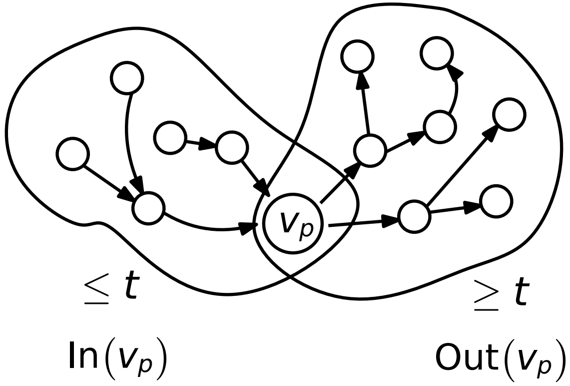

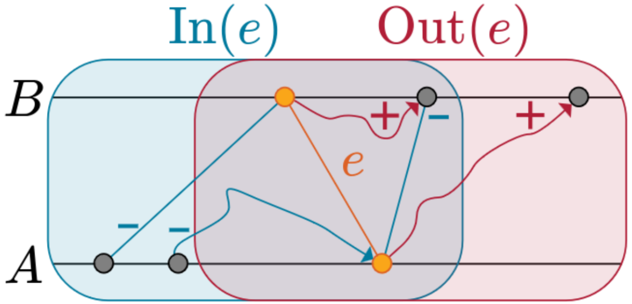

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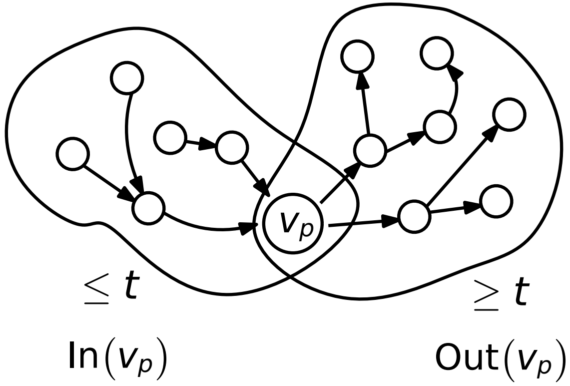

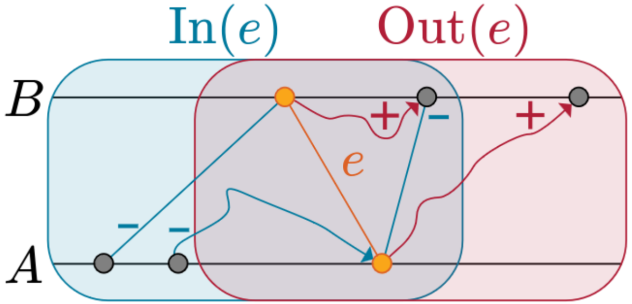

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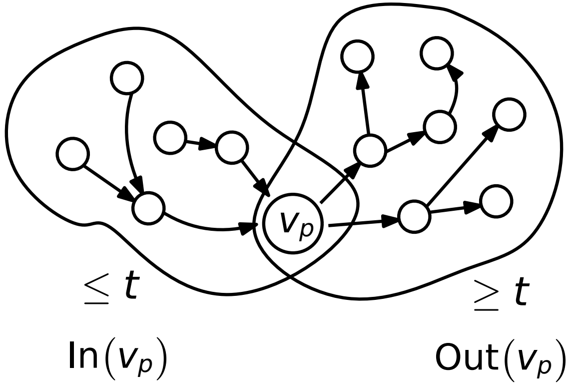

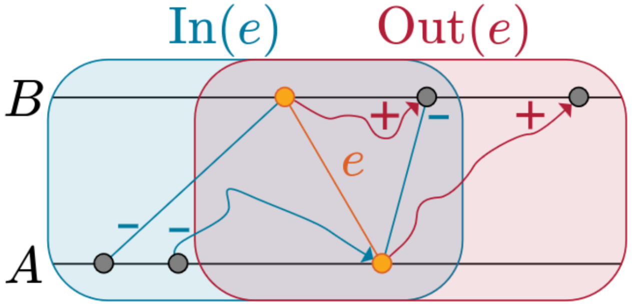

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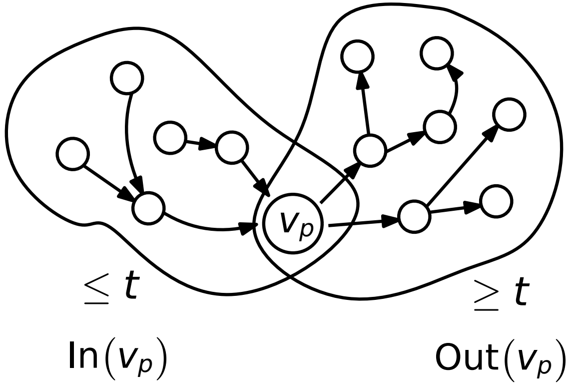

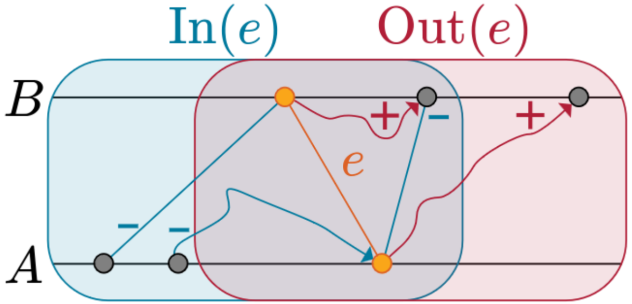

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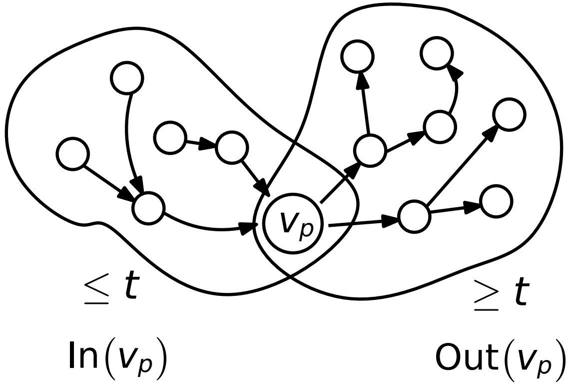

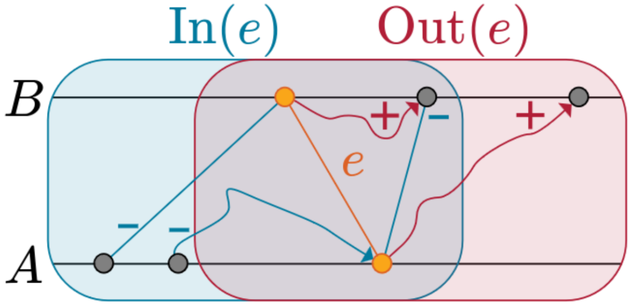

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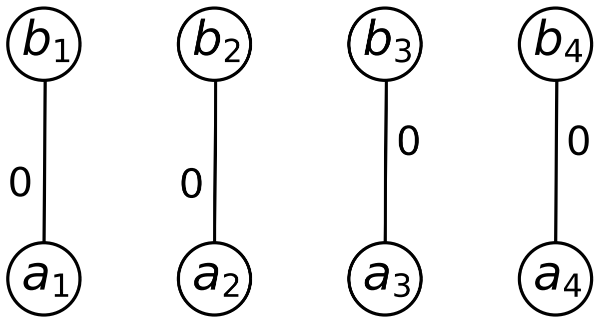

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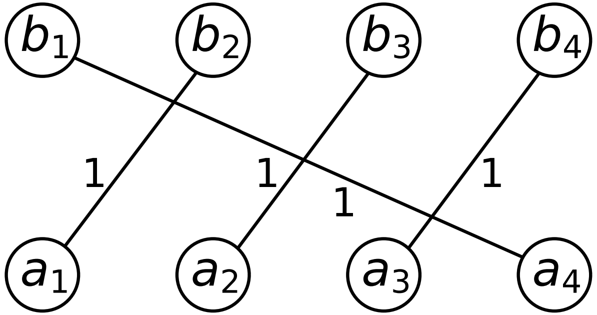

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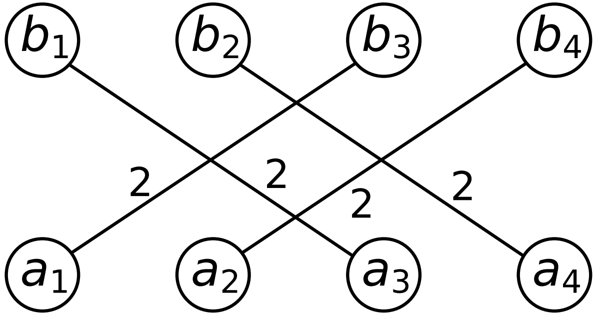

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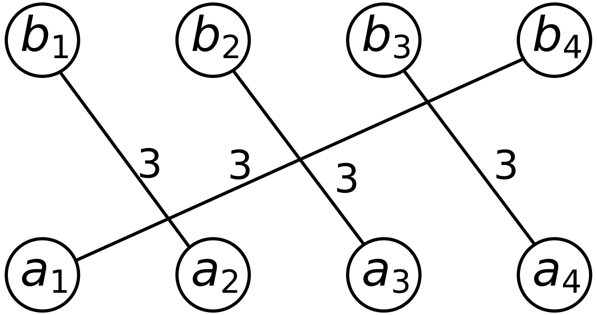

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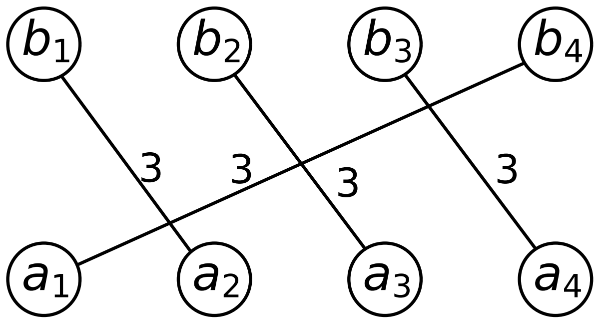

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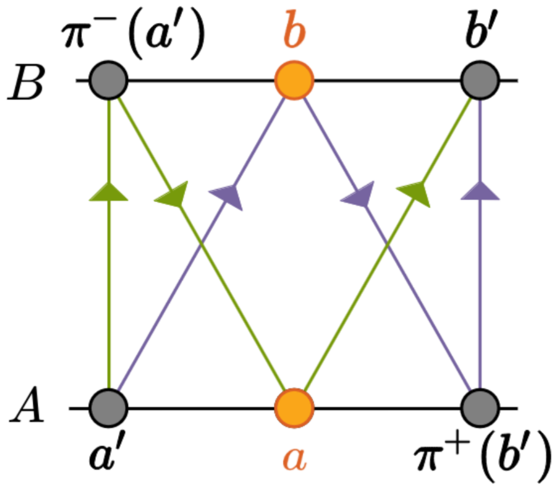

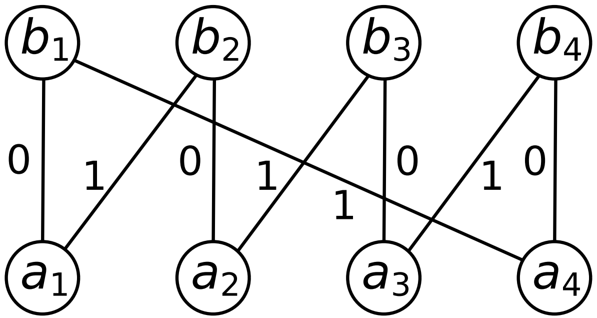

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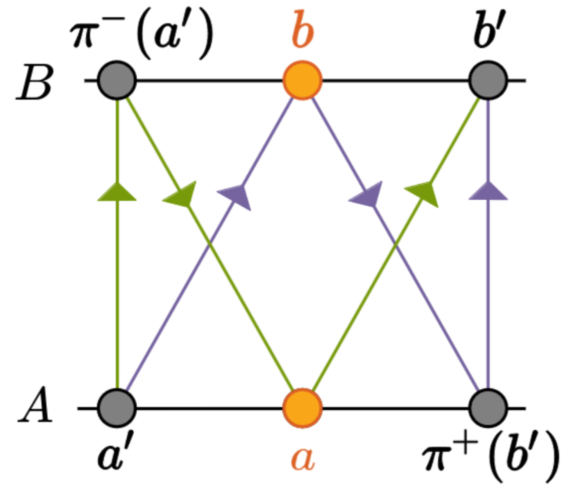

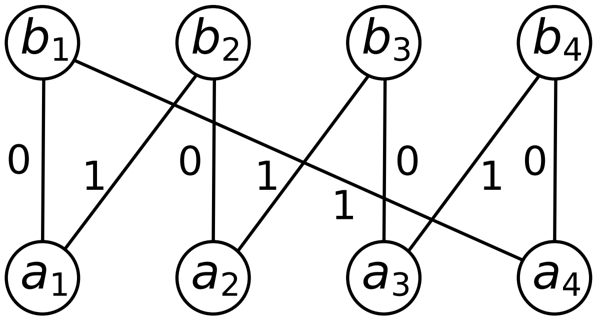

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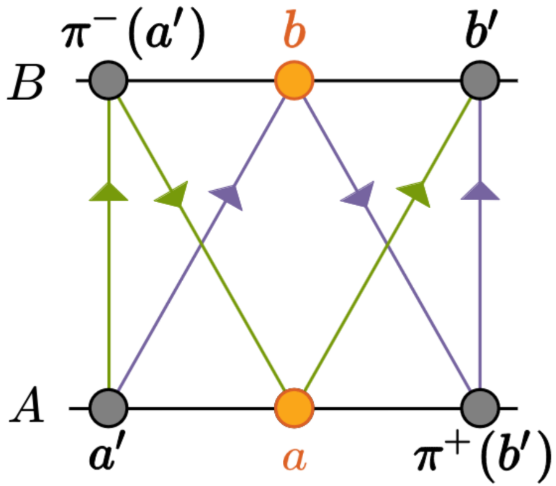

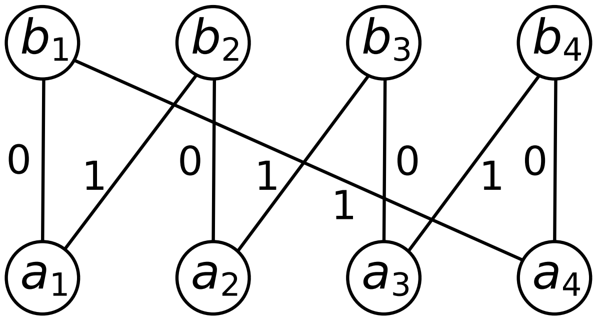

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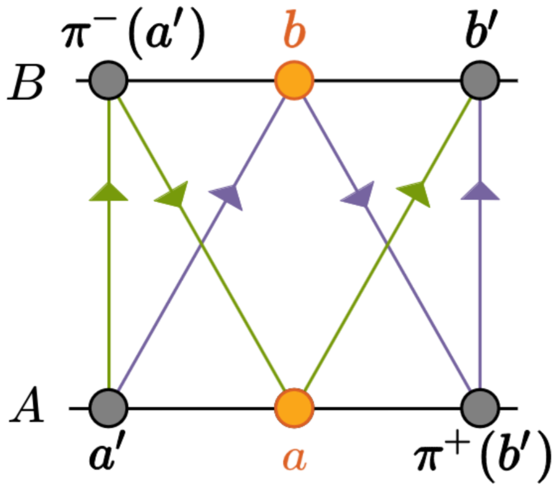

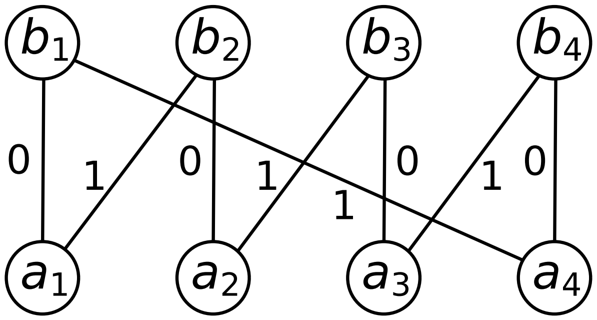

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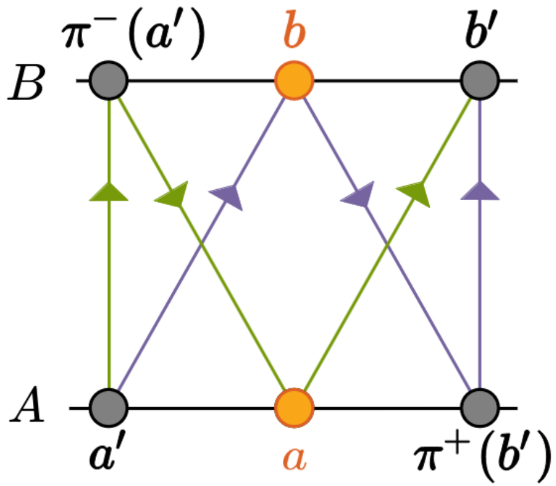

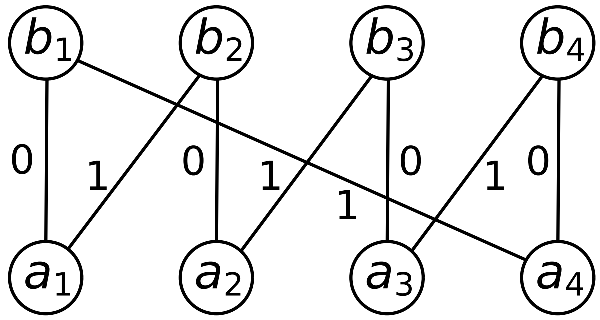

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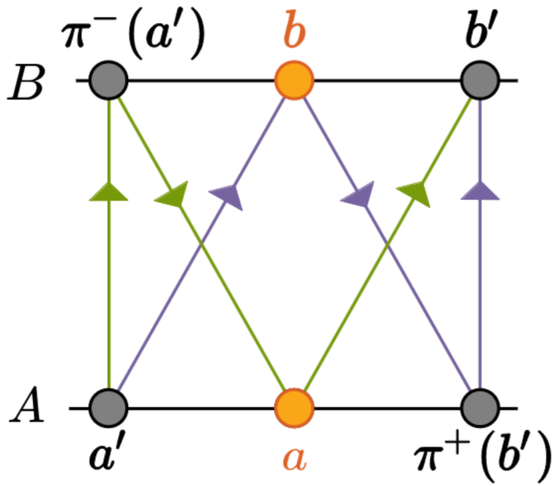
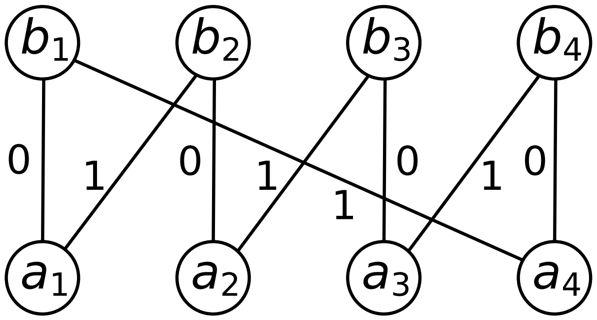
Shifted Matching graph *(SM)*

- 0 or $n - 1$ labelled edges have $\text{NotRev}_e = \emptyset$





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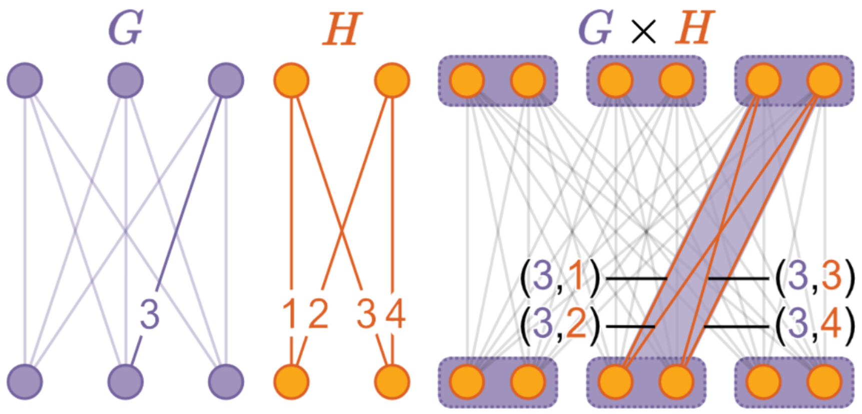

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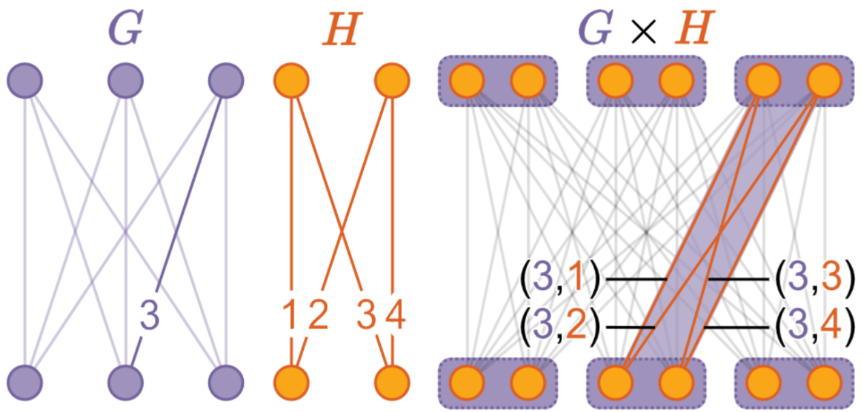

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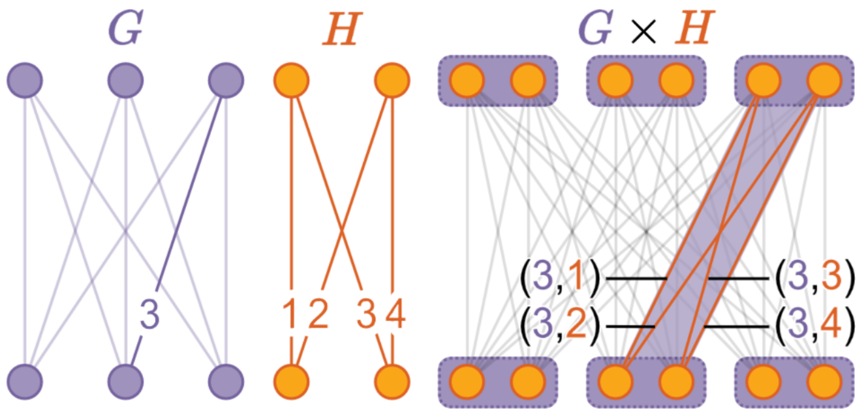
- Compose spanner of size $|S_G| n_H + |S_H| n_G$
  - Also works if each edge is expanded by a different inner graph



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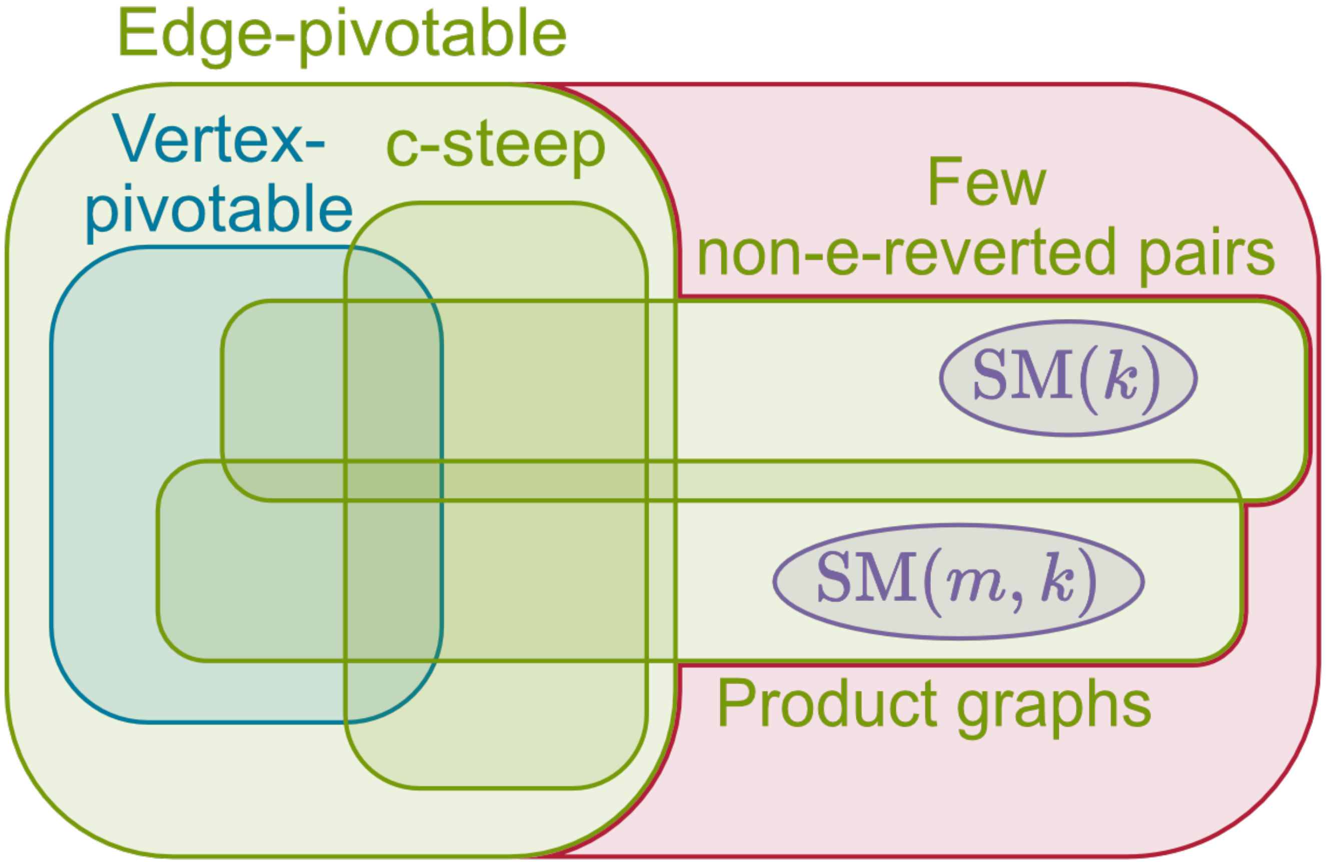
Future Work

- Tweaking structures may give unsolved graphs
- Allow for weaker conditions
- Compose techniques (on a local level)

What did we solve?



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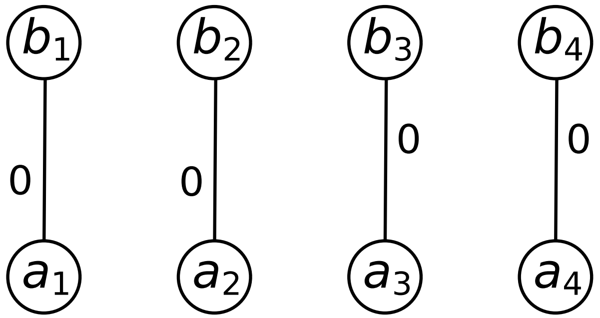

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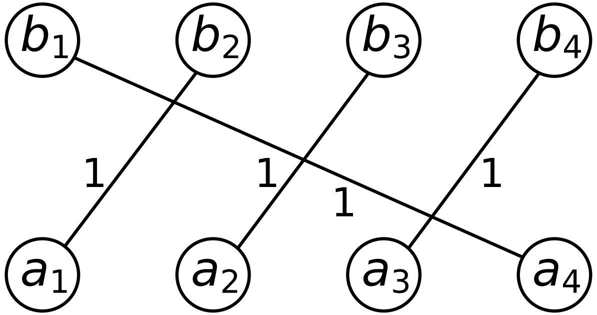

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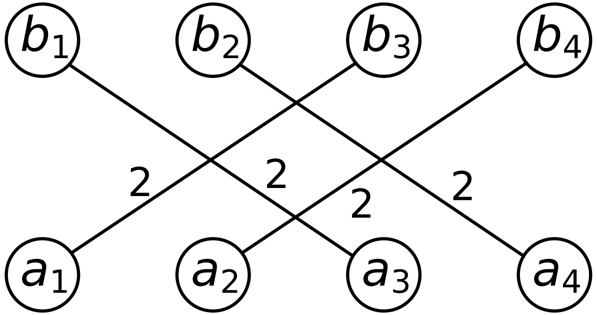

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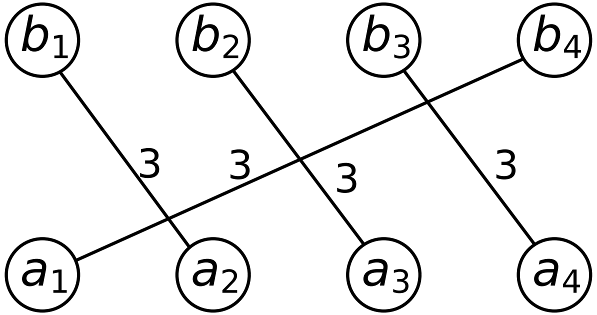

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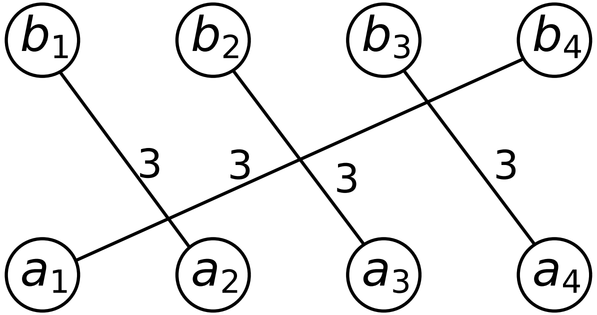

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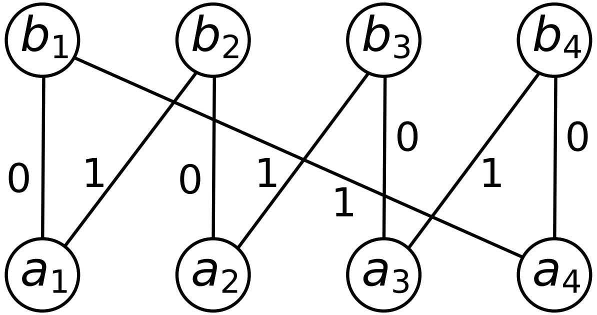

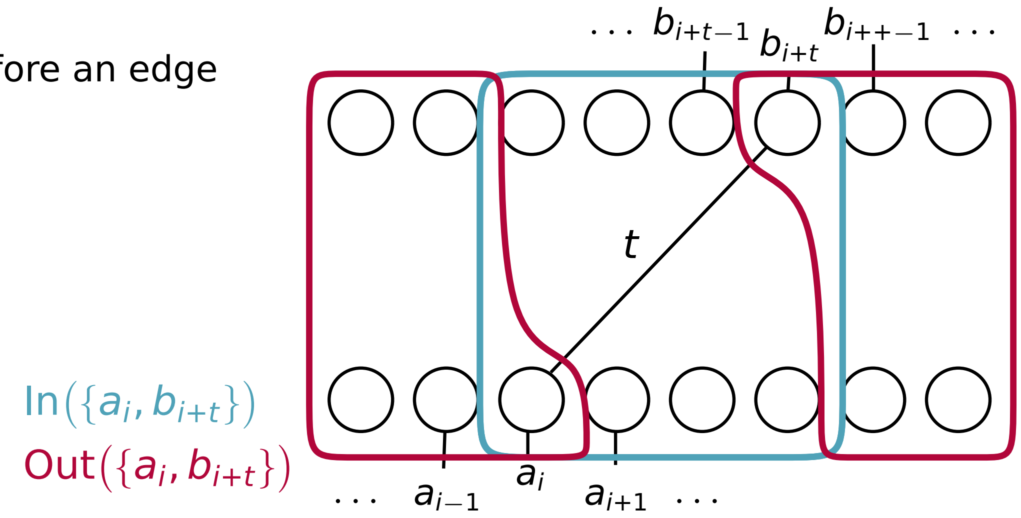

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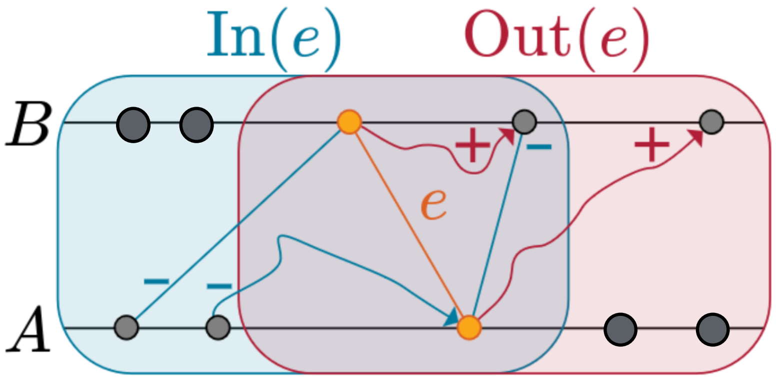

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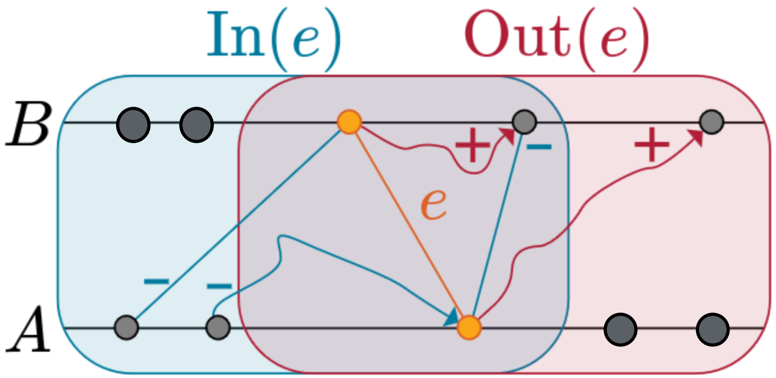

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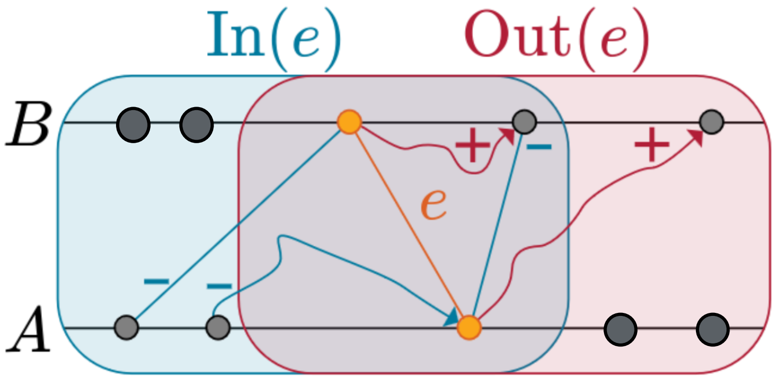
- $\mathrm{In}(e), \mathrm{Out}(e)$ can be connected with $\mathcal{O}(|\mathrm{In}(e)| + |\mathrm{Out}(e)|)$ edges
- There is a spanner of size: $\min_{e \in A \otimes B} \left(\mathcal{D}\left(n - \frac{|\mathrm{In}(e)|}{2}\right) + \mathcal{D}\left(n - \frac{|\mathrm{Out}(e)|}{2}\right) + 2|\mathrm{In}(e)| + 2|\mathrm{Out}(e)| - 3 \right)$
  - Connect $\mathrm{In}(e)$ to $\mathrm{Out}(e)$



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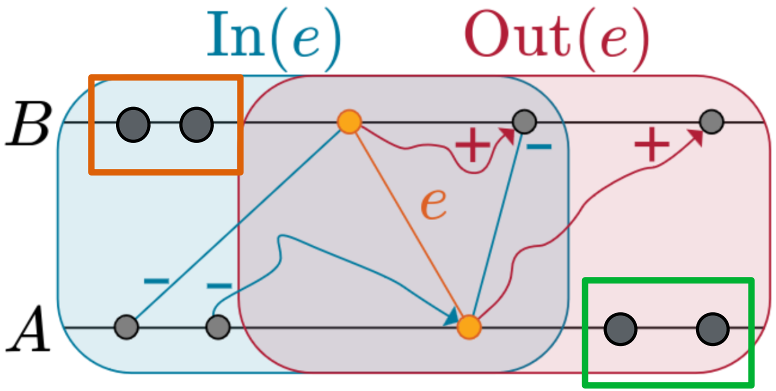

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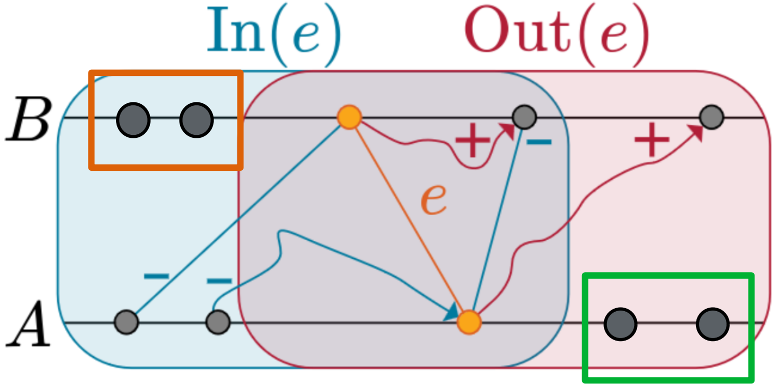

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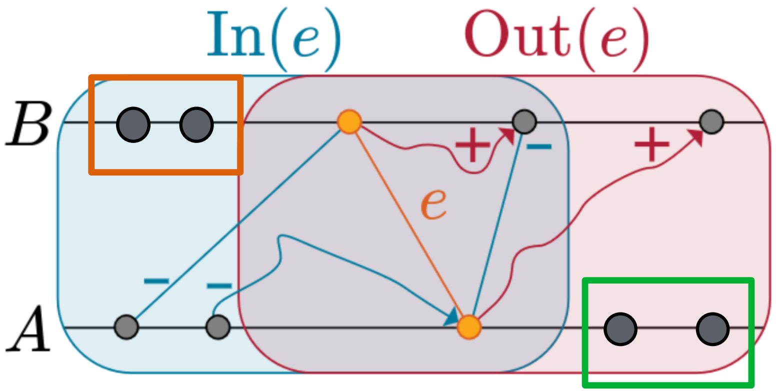
- If $|\mathrm{In}(e) \cap \mathrm{Out}(e)| \in \Omega(n)$, include constant many edges for each reduced vertex
- $\mathrm{In}(e) \cap \mathrm{Out}(e)$ called pivot set of e

$\mathrm{In}(e) \cap \mathrm{Out}(e) \in \Omega(n) \Rightarrow$ reduce!



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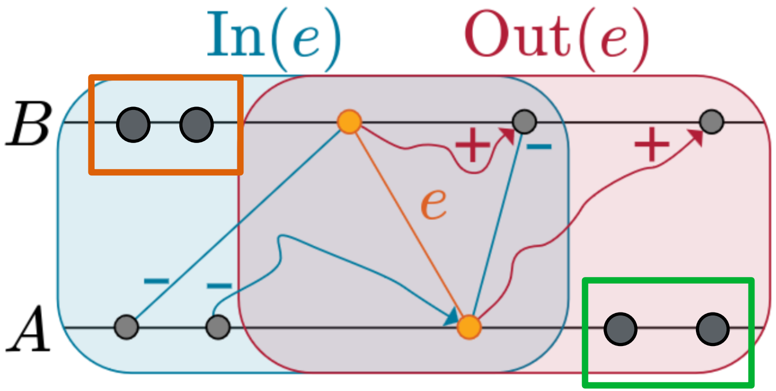

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Is there a graph that has no pivot edges?

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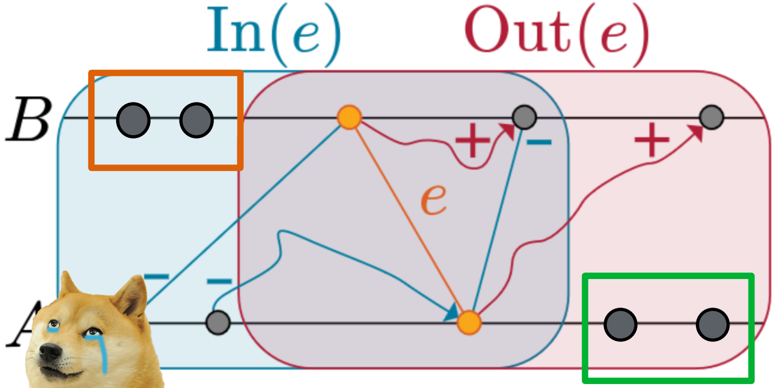

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Is there a graph that has no pivot edges? Yes

Partial Pivot Edges

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Forbidden structures

- Steepness

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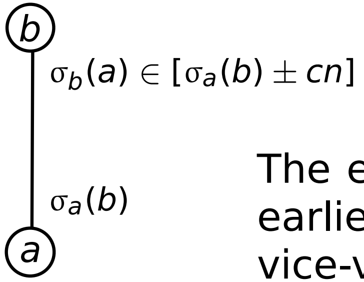
Forbidden structures

- Steepness

$|\{v' \in N(v) \mid i \le \sigma_{v'}(v) \le j\}| < j - i + 2cn$



The edge cannot be much earlier for a than for b (and vice-versa)!

Partial Pivot Edges

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- Steepness
- (Label spread)
- (Activity width)

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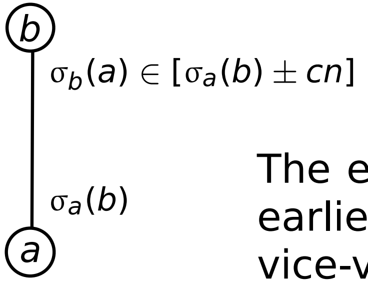

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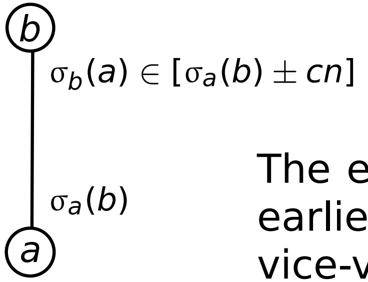

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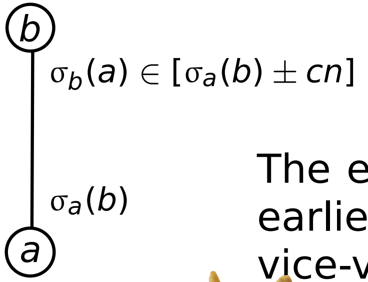


The edge cannot be much earlier for a than for b (and vice-versa)!

Is there a graph that has no pivot edges?

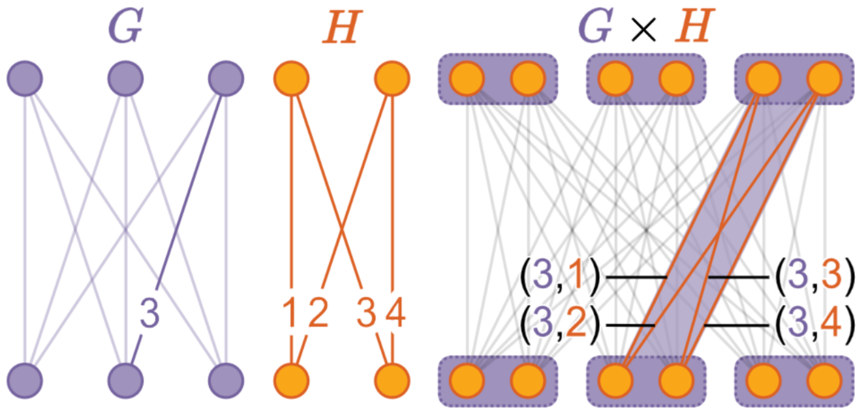
Yes

Composed Graphs—Construction

- Shifted Matchings counterpart for Partial Pivot Edges
- in between: $\log(n)$ or $\sqrt{n}$ pivots / non-reverted edges

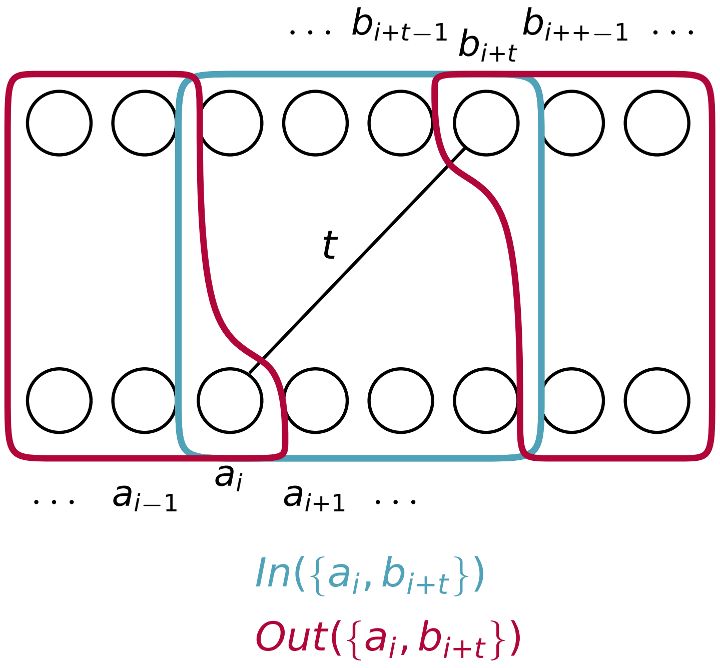
Product graphs

- tensor product of outer (G) and inner (H) graph
- for any $f : \mathbb{N} \to \mathbb{N}^+$ with $f(n) \in \mathcal{O}(n)$, can construct bi-clique s.t. for all $e \in A \otimes B$:
  - $\text{In}(e) \cap \text{Out}(e) \in \mathcal{O}(f(n))$
  - $|\text{NotRev}_e| \in \Omega(n \cdot f(n))$
- can interpolate between parameters



Composed Graphs—Construction

- $SM(m, k)$: outer graph $SM(m)$, inner graph $SM(k)$
  - for any e: $|\,\mathrm{In}(e) \cap \mathrm{Out}(e)| \leq 2k$
  - for any e: $|\,\mathrm{NotRev}_e\,| \geq (m-1)\binom{k}{2}$



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  - for any e: $|\text{In}(e) \cap \text{Out}(e)| \leq 2k$
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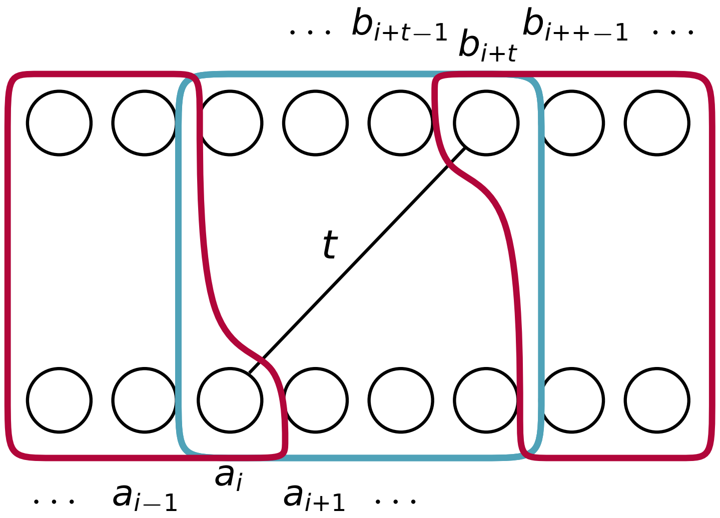
- let $(g_1, h_1)(g_2, h_2) \ldots (g_\ell, h_\ell)$ be temporal
  - $g_1 g_2 \ldots g_\ell$ is temporal



$In(\{a_i, b_{i+t}\})$

$Out(\{a_i, b_{i+t}\})$

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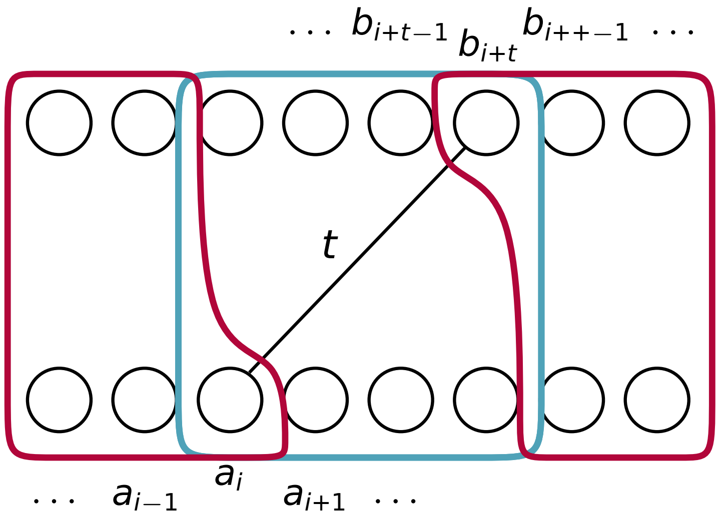
- let $(g_1, h_1)(g_2, h_2)\ldots(g_\ell, h_\ell)$ be temporal
  - $g_1 g_2 \ldots g_\ell$ is temporal
- let $\phi(g, h) = g$ for all $(g, h) \in G \otimes H$
  - $\phi(\text{In}(e)) \subseteq In(\phi(e))$
  - $\phi(\text{In}(e) \cap \text{Out}(e)) \subseteq In(\phi(e)) \cap Out(\phi(e))$
  - $In(\{a_i, b_j\}) \cap Out(\{a_i, b_j\}) = \{a_i, b_j\}$ in SM
  - can only reach two bags



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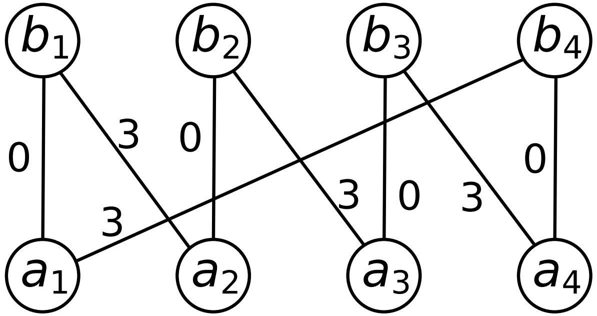

Vertex	Edge Label			
	0	1	2	3
a_0	b_0	b_1	b_2	b_3
a_1	b_1	b_2	b_3	b_0
a_2	b_2	b_3	b_0	b_1
a_3	b_3	b_0	b_1	b_2

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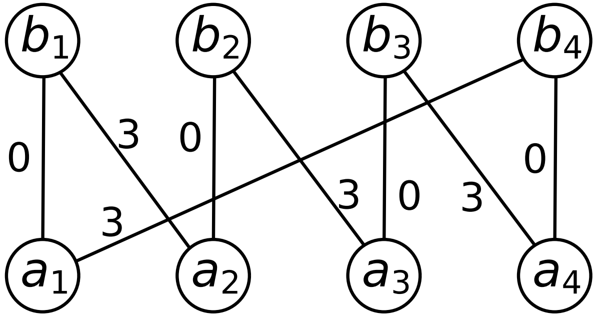

- $e = \{a,b\}$, how many non-e-reverted edges
  - let $a', a'' \in A_\ell$ for $1 \leq \ell \leq m-1$ with $a' \prec_b a''$
  - $\mathrm{NotRev}_e = \{\{a', \pi^+(a'')\} \mid a' \succeq_b a'' \text{ or } \pi^-(a') \succeq_a \pi^+(a'')\}$
  - $\pi^-(a') \in B_\ell$, $\pi^+(a'') \in B_{\ell-1}$, thus $\pi^+(a'') \prec_a \pi^-(a')$
  - $|\mathrm{NotRev}_e| \geq (m-1)\binom{k}{2}$

Vertex	Edge Label			
	0	1	2	3
a_0	b_0	b_1	b_2	b_3
a_1	b_1	b_2	b_3	b_0
a_2	b_2	b_3	b_0	b_1
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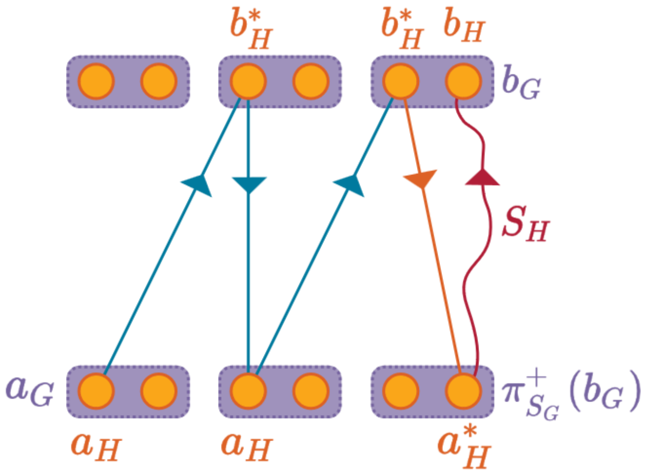
  - $|\text{NotRev}_e| \in \Omega(n \cdot f(n))$
- construct $SM(m, k)$ with $m := 1 + \lceil \frac{n}{f(n)} \rceil$, $k := 1 + f(n)$
  - for any e: $|\text{In}(e) \cap \text{Out}(e)| \leq 2 \cdot (1 + f(n))$
  - $|\text{NotRev}_e| \geq \frac{n}{f(n)} \binom{1+f(n)}{2} \geq \frac{nf(n)}{2}$
  - $m \cdot k = \left(1 + \lceil \frac{n}{f(n)} \rceil\right)(1 + f(n)) \geq \frac{n}{f(n)} \cdot f(n) \in \Omega(n)$
  - $m \cdot k \leq \left(2 + \lceil \frac{n}{f(n)} \rceil\right)(1 + f(n)) \leq 2 + 2f(n) + 2n \in \mathcal{O}(n)$

Composed Graphs—Spanners

- tensor product of outer (G) and inner (H) graph
- compose spanner S of size $|S_G| n_H + |S_H| n_G$



Composed Graphs—Spanners

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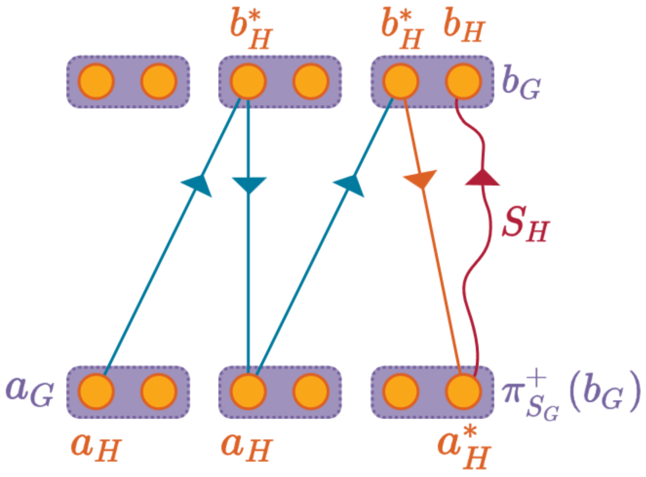
- compose spanner S of size $|S_G|n_H + |S_H|n_G$
  - choose $b^*_H \in B_H, a^*_H \in A_H$ such that $\lambda_H(\{a^*_H, b^*_H\})$ is minimal
  - for $(a_G, b_G) \in S_G$ for all $a_h \in A_H$, add $\{(a_G, a_H), (b_G, b^*_H)\}$ to S
  - for $(a_H, b_H) \in S_H$ for all $b_G \in B_G$, add $\{(\pi^+_{S_G}(b_G), a_H), (b_G, b_H)\}$ to S



Composed Graphs—Spanners

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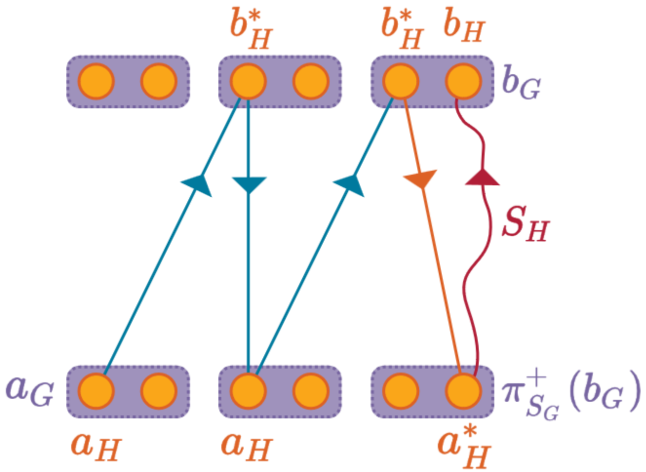
- compose spanner S of size $|S_G|n_H + |S_H|n_G$
  - choose $b_H^* \in B_H, a_H^* \in A_H$ such that $\lambda_H(\{a_H^*, b_H^*\})$ is minimal
  - for $(a_G, b_G) \in S_G$ for all $a_h \in A_H$, add $\{(a_G, a_H), (b_G, b_H^*)\}$ to S
  - for $(a_H, b_H) \in S_H$ for all $b_G \in B_G$, add $\{(\pi^+_{S_G}(b_G), a_H), (b_G, b_H)\}$ to S
- path from $(a_G, a_H) \in A_G \times A_H$ to $(b_G, b_H) \in B_G \times B_H$
  - use path $a_G \to b_G$ from S_G, second last in $(a'_G, a_H) \in A$
  - if $a'_G \neq \pi^+_{S_G}(b_G)$ then next $(b_G, b_H^*), (\pi^+_{S_G}(b_G), a_H^*)$
  - in bags $\pi^+_{S_G}(b_G), b_G$, use spanner S_H



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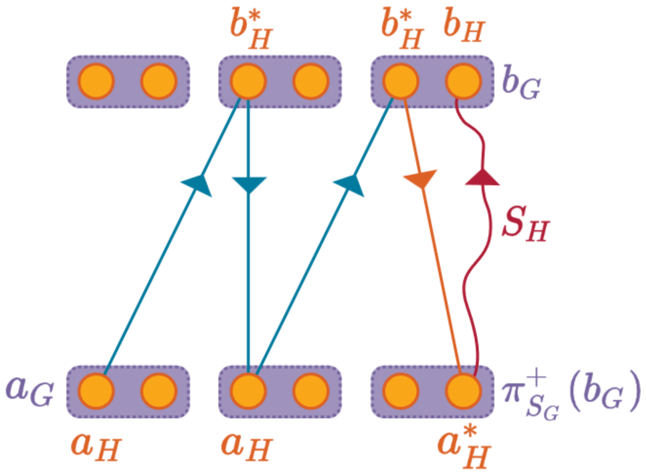

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  - in bags $\pi^+_{S_G}(b_G), b_G$, use spanner S_H
- works for arbitrary composed graphs, adapt b_H^*, a_H^* per bag