Revisiting Diameter in Directed Graphs

ESA 2026

Ben Bals
Centrum Wiskunde & Informatica and VU Amsterdam

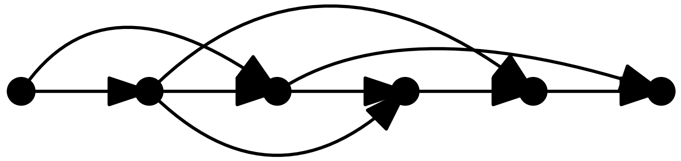
Why Another Diameter Definition?

Option 1: **Classic Diameter**

$$\mathrm{diam}(G) \coloneqq \max_{u,v \in V} d(u,v)$$

Option 2: **Min-Diameter**

$$\mathrm{MinDiam}(G) \coloneqq \max_{u,v \in V} [\min d(u,v), d(v,u)]$$



We look at **Reachability Diameter**

$$\mathrm{ReachDiam}(G) \coloneqq \max_{u,v \in V : u \rightsquigarrow v} d(u,v)$$

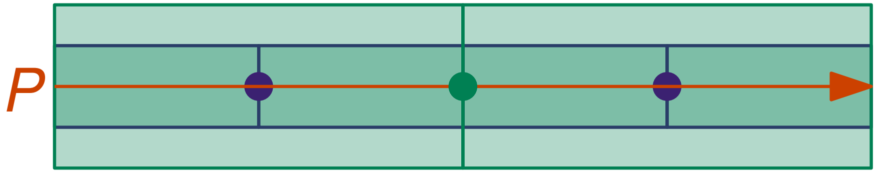
Summary of Previous Work on Classic and Min-Diameter
Small constant approximation algorithms, but none of the techniques transfer

Easy Upper Bounds

Weighted?	Approx.	Det.?	Running time	Technique
✓	exact	✓	$\tilde{O}(nm)$	$n \cdot$ BFS
✓	$1+\varepsilon$	✓	$\tilde{O}(n^\omega/\varepsilon)$	Round + APSP-approx
x	exact	✓	$\tilde{O}(n^\omega)$	Exact transitive closure

Theorem. There is an algorithm computing a $(3, \sqrt{n})$-approximation of ReachDiam in $\tilde{O}(\sqrt{n}m)$ time on unweighted graphs.

- **Part 2: Distance on a chain.** For each $P \in \mathcal{C}$: Run divide & conquer



- Output largest seen distance

Analysis

- Case Diameter path $< \sqrt{n}$: trivially correct

Recall: Any path must hit a vertex from $\mathcal{C}$ after at most $\sqrt{n}$ steps.

Otherwise: must touch one of our chains

- Case Diameter path mostly outside first chain it touches: handled by part 1
- Case Diameter path mostly within first chain it touches: handled by part 2

All Upper Bounds

Shortcut Sets

Weighted?	Approx.	Det.?	Running time	Technique
✓	exact	✓	$\tilde{O}(nm)$	$n \cdot$ BFS
✓	exact	✓	$\tilde{O}(m \cdot \mathrm{tw} + m^{1+o(1)})$	Separator D&C
✓	$1+\varepsilon$	✓	$\tilde{O}(n^{\omega}/\varepsilon)$	Round + APSP-approx
✓	3	✓	$\tilde{O}(m \cdot \lvert \mathrm{MCC} \rvert)$	Path-based approach
x	exact	✓	$\tilde{O}(n^{\omega})$	Exact transitive closure
x	$(1, k)$	x	$\tilde{O}(nm/k)$	Random hitting set
x	$(3, k)$	✓	$\tilde{O}(nm/k)$	ℓ-cover + path-based
x	$n^{1/2+o(1)}$	x	$\tilde{O}(m)$	Sampling pivots

Lower Bounds: Overview

Weighted?	Approx.	Time lower bound	Density	Hypothesis
✓	poly(n)	$n^{\omega - o(1)}$	Dense	SV
✓	poly(n)	$n^{\omega - o(1)}$	Dense	HD-OV
x	< 2	$n^{\omega - o(1)}$	Dense	BMM
x	$< 3/2$	$m^{2 - o(1)}$	Sparse	OV

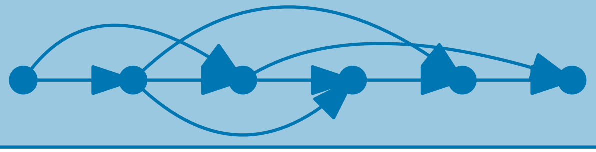
Take-aways

There Are Still Natural, Unstudied Definitions Out There

$\mathrm{diam}(G) \coloneqq \max_{u,v \in V} d(u,v)$

$\mathrm{MinDiam}(G) \coloneqq \max_{u,v \in V}[\min d(u,v), d(v,u)]$

$\mathrm{ReachDiam}(G) \coloneqq \max_{u,v \in V : u \rightsquigarrow v} d(u,v)$



Connections Are Awesome

Shortcut Sets

$\stackrel{?}{\approx}$

ReachDiam

Open Question?

Better apx for unweighted graphs?

Lower Bounds: One Construction

Simplicial Vertex
Is there a vertex whose neighborhood is a clique?

Theorem. In weighted directed graphs, there is no c-approximation of ReachDiam in time $\mathcal{O}(n^{\omega-\varepsilon})$, unless the SV conjecture fails.